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HIGH-FREQUENCY ASYMPTOTIC FORMS OF REFRACTED AND TRANSMITTED WAVES WHEN SOUND IS SCATTERED BY A HOLLOW ELASTIC SPHERE FILLED WITH LIQUID*

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The Sommerfeld-Watson transformation, the reflection method and the theory of graphs are used to investigate the components of the echo-signal from a hollow elastic sphere with a concentric liquid filler, formed by reflection and rereflection at the media interfaces.

A similar approach was used earlier to investigate the sound field in a layer /1/, and the rereflection of waves by a two-layer liquid sphere /2/. Reflected and transmitted (geometric) waves which are of fundamental importance in the potential (background) scattering of sound by an object, were considered in /3—8/ assuming the object to be solid. The presence inside the scatterer of a liquid or elastic filling considerably complicates the process of rereflection /9, 10/. Below, we analyze high-frequency harmonic and non-stationary sound waves reflected and transmitted by a hollow elastic sphere filled with acoustic liquid.

1. The problem is stated and formally solved using a Fourier transformation with respect to time and expansion in spherical harmonics in /11/. The Fourier transform of the pressure in the scattered wave in the form of a Sommerfeld-Watson integral over the complex angular momentum (the normal wave mode number) /12/, which enables the reflected and rereflected components of the echo-signal /2, 3/ to be investigated

$$p_g \approx -ip_0\omega\bar{f}(\omega)\int\limits_C \frac{\nu}{y} R_l h_l^{(1)}(\omega r)\, Q_l^{(2)}(\theta)\, d\nu \qquad (1.1)$$

$$Q_l^{(2)}(\theta)=\frac{1}{2}\left[P_l(\cos\theta)-\frac{2i}{\pi}Q_l(\cos\theta)\right] \quad \left(l=\nu-\frac{1}{2}\right)$$

Here C is the part of the contour which also includes the poles of the function R_l that correspond to surface waves /3, 12—15/ and passes along the path at steepest descent, $P_l(x)$ and $Q_l(x)$ are the Legendre functions, $h_l^{(k)}(x)$ $(k=1, 2)$ are Hankel spherical functions, $\bar{f}(\omega)$ is the Fourier transform of the acoustic signal radiated by a point source at a distance l_0 from the centre of the scatterer, which is taken as the origin of a spherical system of coordinates $r\theta\varphi$ $(\theta=0, \varphi=0$ is the direction to the source), ω is the parameter of the Fourier transformation (the dimensionless frequency), and p_0 is the normalizing pressure. The function R_l is related to x_l by the equation /2, 5/

$$x_l=-\frac{1}{2}\left(1+\frac{1}{y}R_l\right), \quad y=\frac{h_l^{(1)}(\omega)}{h_l^{(2)}(\omega)}$$

where x_l is equal to the ratio of two sixth-order determinants, which follow from the conditions of contact between the elastic and liquid media and composed of combinations of Bessel, Neumann, and Hankel spherical functions /11/. Omitting the details related to the expansion of these determinants, we present the final formula

$$R_l=R_l^\circ-\frac{\bar{f}_3}{(1-f_1)^2}\frac{\bar{y}_F}{1+\bar{R}_l^\circ\bar{y}_F} \qquad (1.2)$$

$$R_l^\circ=R_{12}+\frac{f_2}{1-f_1}, \quad \bar{R}_l^\circ=\bar{R}_{12}+\frac{\bar{f}_2}{1-f_1}$$

$$f_1=\sum_{A,B}(a_AY_A+b_{AB}Y_{LT})-dY_LY_T \qquad (1.3)$$

$$f_2=\sum_{A,B}(e_AY_A+f_{AB}Y_{LT})-gY_LY_T$$

$$\bar{f}_2=-\sum_{A,B}(\bar{e}_AY_A+\bar{f}_{AB}Y_{LT})+\bar{g}Y_LY_T$$

$$\bar{f}_3=-\sum_{A,B}[\bar{h}_AY_A+(\bar{i}_{AB}+\bar{k}_AY_A)Y_{LT}]-\left[\bar{j}+\sum_{A,B}(\bar{n}_AY_A+\bar{l}_{AB}Y_{LT})\right]Y_LY_T$$

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$$a_A=\eta_A\bar{\eta}_A,\quad b_{AB}=m_{AB}r_A\bar{r}_B,\quad d=\delta\bar{\delta}$$
$$e_A=\alpha\lambda_A\bar{\eta}_A,\quad f_{AB}=\alpha m_{AB}q_A\bar{r}_B,\quad g=\alpha\sigma\bar{\delta}$$
$$\bar{e}_A=\bar{\alpha}\eta_A\bar{\lambda}_A,\quad \bar{f}_{AB}=\bar{\alpha}m_{AB}r_A\bar{q}_B,\quad \bar{g}=\bar{\alpha}\delta\bar{\sigma}$$
$$\bar{h}_A=\alpha_0\lambda_A\bar{\lambda}_A,\quad \bar{i}_{AB}=\alpha_0 m_{AB}q_A\bar{q}_B,\quad \alpha_0=\alpha\bar{\alpha}$$
$$\bar{j}=\alpha_0(\varkappa\bar{\varkappa}-\sigma\bar{\sigma}),\quad \bar{k}_A=\bar{k}_{LA}m_{LT}+\bar{k}_{TA}m_{TL}$$
$$\bar{k}_{AB}=\alpha_0\mu_{AB}\bar{\mu}_{BA},\quad \bar{k}_{AA}=\alpha_0\mu_{AA}\bar{\mu}_{BB}$$
$$\bar{l}_{AB}=\alpha_0 m_{AB}\xi_A\bar{\xi}_A,\quad \bar{n}_A=\alpha_0\chi_A\bar{\chi}_A$$
$$\lambda_A=\lambda_{ij}{}^{22},\quad \chi_A=\lambda_{ij}{}^{11},\quad \varkappa=\lambda_{12}{}^{21},\quad \sigma=\lambda_{22}{}^{11},\quad \delta=\eta_{11}$$
$$q_A=\xi_A{}^{22},\quad \xi_A=\xi_A{}^{11},\quad \mu_{AB}=-\xi_A{}^{ji},\quad \eta_A=\eta_{ij}$$
$$\bar{\lambda}_B=\bar{\lambda}_{ij}{}^{11},\ \bar{\chi}_B=\bar{\lambda}_{ij}{}^{22},\quad \bar{\varkappa}=\bar{\lambda}_{12}{}^{21},\quad \bar{\sigma}=\bar{\lambda}_{11}{}^{22},\quad \bar{\delta}=\bar{\eta}_{22}$$
$$\bar{q}_A=\bar{\xi}_A{}^{11},\quad \bar{\xi}_A=\bar{\xi}_A{}^{22},\quad \bar{\mu}_{AB}=-\bar{\xi}_A{}^{ji},\ \bar{\eta}_B=\bar{\eta}_{ij}$$
$$\lambda_{pq}{}^{kt}=(A_{kt}B_{pq}-B_{kt}A_{pq})\,C_s{}^{-1},\ \xi_A{}^{kt}=(A_{kt}F_{2A}-B_{kt}F_{1A})\,C_s{}^{-1}$$
$$\eta_{pq}=(z_1B_{pq}-N_sA_{pq})\,C_s{}^{-1},\ r_A=(z_1F_{2A}-N_sF_{1A})\,C_s{}^{-1}$$
$$\alpha=(z_2-z_1)\,N_sC_s{}^{-1},\ \bar{\alpha}=(\bar{z}_{2F}-\bar{z}_{1F})\,N_F\bar{C}_F{}^{-1}$$
$$R_{12}=(z_2B_{22}-N_sA_{22})\,C_s{}^{-1},\ \bar{R}_{12}=(\bar{z}_{1F}\bar{B}_{11}-N_F\bar{A}_{11})\,\bar{C}_F{}^{-1}$$
$$C_s=z_1B_{22}-N_sA_{22},\ \bar{C}_F=\bar{z}_{2F}\bar{B}_{11}-N_F\bar{A}_{11}$$
$$A_{kt}=z_{kL}s_{tT}-w_{kT},\ B_{kt}=v_{kL}s_{tT}-u_{kL}w_{tT}$$
$$F_{1L}=F_{1L}{}',\ F_{1T}=l_1F_{1T}{}',\ F_{2A}=[1-(l_1-2)\,q_0]F_{1A}$$
$$F_{1A}{}'=q_0\,(z_{2A}-z_{1A}),\ l_1=l\,(l+1),\ q_0=2\omega_T{}^{-2}$$
$$u_{kA}=q_0\,(1-z_{kA}),\ \ v_{kA}=1-q_0\,(l_1-2z_{kA})$$
$$w_{kA}=l_1u_{kA},\ s_{kA}=u_{kA}+v_{kA}$$
$$z_k=\omega\frac{h_l^{(k)'}(\omega)}{h_l^{(k)}(\omega)},\quad z_{kA}=\omega_A\frac{h_l^{(k)'}(\omega_A)}{h_l^{(k)}(\omega_A)}$$
$$y_A=t_At_A{}^{*-1},\ Y_A=y_Ay_A{}^{-1},\ Y_{LT}=-m_{LT}a_{TL},\ m_{LT}=|\,a_{LT}\,|$$
$$a_{LT}=t_L\bar{t}_T{}^*\,(t_T\bar{t}_L{}^*)^{-1},\ t_A=h_l{}^{(1)}\,(\omega_A),\ \ N_s=\rho\rho_s{}^{-1},\ \ N_F=\rho_F\rho_s{}^{-1}$$
$$(A,\ B=L,\ T;\ A\neq B;\ i,\ j,\ k,\ p,\ q,\ t=1,\ 2;\ i\neq j)$$

where the asterisk denotes complex conjugation. It is also assumed that when $A=L$, $B=T$, then $i=1$, $j=2$, and when $A=T$, $B=L$, then $i=2$, $j=1$. In formulas with a bar all functions depend on $\bar{\omega}_A=\omega_A\varepsilon$, where $\omega_A=\omega\beta_A{}^{-1}$, $\beta_A=c_Ac^{-1}$ $(A=L,\ T,\ F)$. In this case z_1, N_s and C_s are replaced by $\bar{z}_{2F}$, N_F and $\bar{C}_F$. Here $\varepsilon=ba^{-1}$ is the ratio of the internal and external radii of the shell, c_L, c_T, ρ_s are the velocities of the longitudinal and transverse waves, and the density of the shell material, c, c_F, ρ, ρ_F are the velocities of sound and the densities of the external acoustic medium and the filler, and $h_l^{(k)'}(x)$ is the derivative of the Hankel spherical function with respect to the argument. The introduction of phase functions (the ratio of the Hankel functions of the first and second kind) and the logarithmic derivative of the Hankel functions is convenient for a geometric description of the wave propagation. Thus the first term in R_l° in (1.2) describes the wave reflection from the external surface of the shell, and the second describes the propagation in the shell layer without entering the filler. A similar wave picture on the side of liquid filler is obtained by considering $\bar{R}_l^\circ$. The second term in R_l (1.2) corresponds to rereflection of waves in the shell on passing into the filler.

In a special case, by passing to the limit $\varepsilon\to 0$ we obtain an expression for R_l that corresponds to the scattering of sound by a solid elastic sphere

$$R_l\equiv R_l^\circ=R_{12}+\frac{f_2}{1-f_1},\quad f_1=\eta_Ly_L+\eta_Ty_T+\delta y_Ly_T$$
$$f_2=-\alpha\,(\lambda_Ly_L+\lambda_Ty_T+\sigma y_Ly_T)$$

As $c_T\to 0$ we obtain the case of a liquid two-layer sphere /2/. If $\varepsilon\to 0$ or $\varepsilon\to 1$, we obtain the case of a dense liquid sphere with parameters c_L, ρ_s or c_F, ρ_F, respectively.

2. For further analysis it is important to establish the physical meaning of the coefficients in R_l (1.2), and to determine their connection with the reflection and transmission coefficients. For a solid elastic cylinder this was done in /5/ using complicated transformations. The same problem was solved more simply by a graphical method in /1/. * By

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*Poddubnyak, A.P., On a method of determining geometric waves, when an acoustic wave is scattered by an elastic circular cylinder. In: Proceedings of the 6-th Conference of Young Scientists. Inst. Appl. Mech. and Math. Academy of Sciences of the UkrSSR, L'vov, 1978.

virtue of the presence of an additional inner interface in the object, the process of wave rays splitting in the form of a multicomponent chain reaction becomes more complicated. Here the theory of graphs is particularly advantageous.

We will write the function R_l in a form equivalent to (1.2)

$$R_l = R_{12} + \frac{\alpha\left[\sum_{A,B}(\lambda_A \bar{S}_A Y_A - q_A \bar{Q}_{AB} Y_{LT}) - \sigma \bar{S}_{LT} Y_L Y_T\right]}{1 - \left[\sum_{A,B}(\eta_A \bar{S}_A Y_A - r_A \bar{Q}_{AB} Y_{LT}) - \delta \bar{S}_{LT} Y_L Y_T\right]} \tag{2.1}$$

$$\bar{S}_A = \bar{\eta}_A + \frac{\bar{\alpha}\bar{\lambda}_A \bar{y}_F}{1 + \bar{R}_{21}\bar{y}_F}, \quad \bar{S}_{LT} = \bar{\delta} + \frac{\bar{\alpha}\bar{\sigma}\bar{y}_F}{1 + \bar{R}_{21}\bar{y}_F} \tag{2.2}$$

$$\bar{Q}_{AB} = m_{AB}\left(\bar{r}_A + \frac{\bar{\alpha}\bar{q}_A \bar{y}_F}{1 + \bar{R}_{21}\bar{y}_F}\right), \quad A, B = L, T; A \neq B$$

We will represent each ray of the longitudinal and transverse wave by continuous and dashed lines, considering them to be the edges of the graph, and the points of ray splitting as the boundary points of the edge (the vertices of the graph). Four graphs are shown in Fig.1 in which the small circles denote the vertices of the graph on the external surface of the shell $r = 1$, while the black dots represent the vertices of the graph on the internal surface of the shell $r = \varepsilon$. The graphs are constructed for the sums of all L- and T- modes in the shell passing through the surface $r = 1$ into the external acoustic medium, related to the transmission coefficient and the phase path of the respective wave

$$\overset{\circ}{M}_{jA} = \frac{M_{jA}}{X_A T_{12}{}^A} \quad (A = L, T; j = 1, 2) \tag{2.3}$$

Cutting any graph at its vertex we obtain three parts, one of which corresponds to the "exciting" wave, and the other two to the "excited" L- and T- modes which are of the same type as the initial ones.

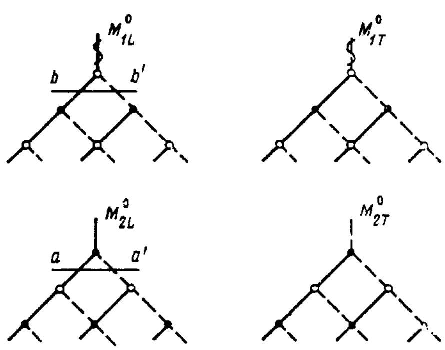

Fig.1

In Fig.1 and in (2.3) M_{2A}° denotes an A-graph ($A = L, T$) with waves formed as a consequence of sound waves penetrating into the shell, and their reflection and splitting into L- and T- modes on the surface $r = \varepsilon$ and their emergence into the external medium, or after rereflection on the external and internal shell surfaces followed by transmission into the external acoustic medium. In the case of M_{1A}° we have an A-graph with waves formed after rereflection from the external surface of the object $r = 1$ from inside the back into the shell thickness, followed by rereflection from the internal surface and emergence into the external acoustic medium. If the penetrating rays, at least L- mode waves, do not touch the filler surface, the graph M_{2L}° corresponds to the case of solid sphere. The vertices of the graph M_{2L}° and of the graph M_{1T}° are on the surface $r = 1$ (all points in Fig.1 of these graphs must be black dots). We do not alter graph M_{2T}° and the corresponding graphs M_{2L}°, M_{1L}°, M_{1T}°. It is only, when the L- and T- modes propagate outside the filler, that we leave only two graphs M_{2L}° and M_{2T}° with black dots substituted for all points.

The cut aa' on graph M_{2L} (Fig.1) yields below it the sum of graphs M_{1L}° and M_{1T}° multiplied by phase functions of the transmission paths of the L- and T- modes and the reflection coefficients of the internal surface $r = \varepsilon$

$$\overset{\circ}{M}_{2L} = F^L + \bar{R}_{12}^{LLF}\overset{\circ}{M}_{1L}X_L + \bar{R}_{12}^{LTF}\overset{\circ}{M}_{1T}X_T \tag{2.4}$$

where F^L denotes rays which first emerge after a single internal reflection and splitting on surface $r = \varepsilon$ (a single L-ray in the case of solid)

$$F^L = -\bar{R}_{12}^{LLF}T_{21}{}^L X_L - \bar{R}_{12}^{LTF}T_{21}{}^T X_T \tag{2.5}$$

where $\bar{R}_{12}{}^{LAF}$ are the values of the reflection coefficients characterizing the wave process in the acoustic filler, $T_{21}{}^A$ are the transmission coefficients of the corresponding waves passing from the shell into the external acoustic medium, X_L and X_T are phase functions that have the property (in the asymptotic sense for high $|\omega|$)

$$X_A^2 = -Y_A, \; Y_{LT} = -X_L X_T \tag{2.6}$$

The cut bb' of the graph M_{1A}° (Fig.1) similarly yields the sum

$$M_{1L}^{\circ} = R_{21}^{LL} M_{2L}^{\circ} X_L + R_{21}^{LT} M_{2T}^{\circ} X_T \tag{2.7}$$

where $R_{21}{}^{LA}$ are the reflection coefficients of waves at the surface $r = 1$ from the inside and back to the shell.

Interchanging the subscripts L and T, in (2.4) and (2.7) we obtain analogous expressions for M_{2T}° and M_{1T}°. As the result we obtain a system of fourth-order linear algebraic equations for M_{jA}°. Solving that system using (2.3) we find M_{2L} and M_{2T}. Summing all waves first rereflected in the shell, and then propagating into the external acoustic medium, we obtain

$$U = M_{2L} + M_{2T} \tag{2.8}$$

Formula (2.1) may be represented in the form *

$$R_l = R_{12} - U \tag{2.9}$$

where R_{12} is the reflection coefficient of the external surface of the body. We equate terms of like phase functions Y in the numerators and denominators of the second terms of the right-hand sides of (2.1) and (2.9). Taking into account that the structure of the coefficients $\bar{R}_{12}^{ABF}$ is of the same form as (2.9) for a liquid sphere (which is also established using the graphs)

$$\bar{R}_{12}^{ABF} = \bar{R}_{12}^{AB} - \bar{U}^{AB}, \quad \bar{U}^{AB} = \frac{\bar{T}_{12}{}^A \bar{y}_F \bar{T}_{21}{}^B}{1 + \bar{R}_{21} \bar{y}_F}$$

where $\bar{R}_{21}$ is the reflection coefficient of the wave from the inner surface of the shell into the filler, $\bar{T}_{12}{}^A$ is the transmission coefficient of an A-mode ($A = L, T$) from the shell to the filler, $\bar{T}_{21}{}^A$ is the transmission coefficient of the sound wave passing from the acoustic filler to the shell when it is converted into an A-mode wave ($A = L, T$), we obtain

$$\begin{aligned}
&\eta_A = R_{21}^{AA}, \quad \sqrt{m_{AB}}\, r_A = R_{21}^{BA}, \quad \alpha\lambda_A = T_{12}{}^A T_{21}{}^A \\
&\sqrt{m_{AB}}\, \alpha q_A = T_{12}{}^A T_{21}{}^B, \quad \delta = \eta_L \eta_T - r_L r_T \\
&\sigma = \sum_{A,B} (\lambda_A \eta_B - r_A q_B), \quad \bar{\eta}_A = -\bar{R}_{12}^{AA} \\
&\sqrt{m_{AB}}\, \bar{r}_B = \bar{R}_{12}{}^{AB}, \quad \bar{\alpha}\bar{\lambda}_A = \bar{T}_{12}{}^A \bar{T}_{21}{}^A \\
&\sqrt{m_{AB}}\, \bar{\alpha}\bar{q}_B = -\bar{T}_{12}{}^A \bar{T}_{21}{}^B, \quad \bar{\delta} = \bar{\eta}_L \bar{\eta}_T - \bar{r}_L \bar{r}_T \\
&\bar{\sigma} = \sum_{A,B} (\bar{\lambda}_A \bar{\eta}_B - \bar{r}_A \bar{q}_B)
\end{aligned} \tag{2.10}$$

Relations for the other coefficients in (1.3) are established by comparing (1.2) with (2.1).

Thus by using graphs a physical meaning can be given to the coefficients in (1.3) and (2.1), and they can be expressed in terms of the reflection and transmission coefficients of the two spherical interfaces $r = 1$ and $r = \varepsilon$.

3. We will evaluate the Sommerfeld-Watson integral using the method of steepest descent /16/, assuming that $|\omega|$ and $|\nu|$ are large. For this we use the respective asymptotic formulas for the Hankel spherical functions and Legendre functions /11, 12/ appearing in (1.1)−(1.3). From these asymptotic forms and (2.6) we represent R_l in the form of an expansion in reflected and rereflected waves. The result of evaluation of the integral is

$$\bar{p}_g \approx \frac{1}{r} p_0 \bar{f}(\omega) \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \sum_{s=0}^{\infty} S_{nms} R_{nms} (N_{nms}) \exp(i\omega L_{nms}^{+}) \tag{3.1}$$

In the case of a high-frequency short-duration sinusoidal signal /6−8/, we have

$$f(\tau) = \sin \omega_0 \tau \, [U(\tau) - U(\tau - \tau_0)], \quad \tau = \frac{ct}{a}, \quad \tau_0 = \frac{ct_0}{a}$$

where ω_0 is the carrier frequency, t is the time, and t_0 is the signal duration. The components of the geometric part of the echo-signal yield

*See the previous footnote.

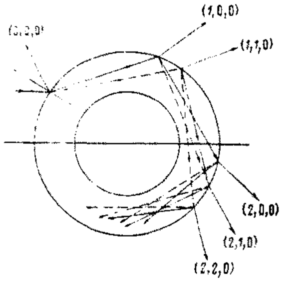

Fig.2

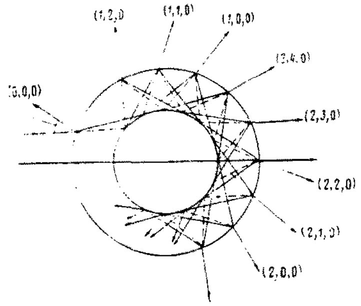

Fig.3

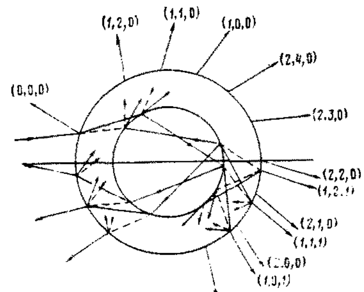

Fig.4

$$p_g \approx \frac{1}{r} p_0 \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \sum_{s=0}^{\infty} S_{nms} R_{nms} (\overset{\circ}{N}_{nms}) \times f(\tau - L^+_{nms}), \quad \overset{\circ}{N}_{nms} = N_{nms}(\omega_0) \tag{3.2}$$

Here $N_{nms}(\omega) \equiv N = \nu/\omega = \sin\gamma$ are the real solutions of the saddle-point equation /16/

$$\begin{aligned}\theta_{nms} = {} & 2\gamma - \varkappa_0 - \varkappa_1 - (2n - m)(\gamma_L - \bar{\gamma}_L) - m(\gamma_T - \bar{\gamma}_T) - \\ & 2s\bar{\gamma}_F + \frac{\pi}{2}\{mU(1 - \beta_T N) + (2n - m)[U(1 - \beta_L N) - \\ & U(\varepsilon - \beta_L N)] + (2s - m) U(\varepsilon - \beta_T N)\}\end{aligned} \tag{3.3}$$

$$\begin{aligned}\sin\gamma = r\sin\varkappa_1 = l_0 \sin\varkappa_0 = \beta_A^{-1}\sin\gamma_A = \\ \varepsilon\beta_B^{-1}\sin\bar{\gamma}_B \; (A = L, T;\; B = L, T, F)\end{aligned}$$

$$S_{nms} = \frac{1}{\cos\varkappa_0 \cos\varkappa_1} \left(\frac{\sin\gamma}{\sin\theta\, |L^-_{nms}|}\right)^{1/2} \exp\left(i\frac{\pi}{4} \overset{\circ}{f}_{nms}\right) \tag{3.4}$$

$$\begin{aligned}L^{\pm}_{nms} = {} & (l_0\cos\varkappa_0)^\alpha + (r\cos\varkappa_1)^\alpha - 2(\cos\gamma)^\alpha + (2n - m)[(\beta_L^{-1}\cos\gamma_L)^\alpha - \\ & (\varepsilon\beta_L^{-1}\cos\bar{\gamma}_L)^\alpha] + m[(\beta_T^{-1}\cos\gamma_T)^\alpha - (\varepsilon\beta_T^{-1}\cos\bar{\gamma}_T)^\alpha] + \\ & 2s(\varepsilon\beta_F^{-1}\cos\bar{\gamma}_F)^\alpha, \quad \alpha = \pm 1/2\end{aligned}$$

$$\overset{\circ}{f}_{nms} = -3 + \operatorname{sign}(L^-_{nms}) + f^1_{nms}$$

$$\begin{aligned}f^1_{nms} = {} & m[U(N\beta_L - 1)\, U(\beta_T - \varepsilon\beta_L) + U(N\beta_T - \varepsilon) \times \\ & U(\varepsilon\beta_L - \beta_T) - U(N\beta_T - 1)] + 2[nU(\beta_T - \varepsilon\beta_L) + \\ & sU(\varepsilon\beta_L - \beta_T)][U(N\beta_T - \varepsilon) - U(N\beta_L - 1)] + \\ & (2n - m - 2s)[U(N\beta_L - \varepsilon) - U(N\beta_T - \varepsilon)\, U(\beta_T - \\ & \varepsilon\beta_L) - U(N\beta_L - 1)\, U(\varepsilon\beta_L - \beta_T)] - 2sU(\varepsilon - N\beta_T)\end{aligned}$$

Here and below, $U(x)$ and $U_{\pm}(x)$ are the symmetric and antisymmetric unit functions.

For the functions R_{nms} we have

$$R_{000} = R_{12}, \quad R_{nm0} = \overset{\circ}{B}_{nm}, \quad R_{nms} = -B^s_{nm} \; (n, m, s \neq 0) \tag{3.5}$$

The coefficient $\overset{\circ}{B}_{nm}$ corresponds to rereflection from the shell surface and refraction back to the external medium, while B^s_{nm} corresponds to transmission into the filler, rereflection on the filler surface and refraction into the external acoustic medium.

The ray pattern of reflection, refraction and rereflection of geometric waves is shown in Figs.2–4 for various angles of incidence of the sound pulse on the shell surface. The notation corresponds here to the subscripts (nms) of the amplitude functions R_{nms} in (3.5).

For $\overset{\circ}{B}_{nm}$, B^s_{nm} as a function of $\overset{\circ}{N}_{nms}$ we obtain the following dependence on the reflection and transmission coefficients in the shell thickness and the filler (2.10):

$$\begin{aligned}\overset{\circ}{B}_{nm} = {} & (e_L \overset{\circ}{A}_{n-1,m} + g\overset{\circ}{A}_{n-2,\,m-2})_{m\neq\{2n-1,\,2n\}} + e_T \overset{\circ}{A}_{n-1,\,m-2} - \\ & f(\overset{\circ}{A}_{n-1,\,m-1})_{m\neq 2n}\end{aligned} \tag{3.6}$$

$$\begin{aligned}B^s_{nm} = {} & (\bar{h}_L W^s_{n-1,\,m} - \bar{j}W^s_{n-2,\,m-2} - \bar{n}_T W^s_{n-3,\,m-4})_{m\neq\{2n-1,2n\}} + \\ & \bar{h}_T W^s_{n-1,\,m-2} - \bar{n}_L (W^s_{n-3,\,m-2})_{m\neq\{2n-3,\,2n-2,\,2n-1,\,2n\}} + \\ & (\bar{i}W^s_{n-1,\,m-1} + \bar{k}_T W^s_{n-2,\,m-3})_{m\neq 2n} + (\bar{k}_L W^s_{n-2,\,m-1} + \\ & \bar{l}W^s_{n-3,\,m-3})_{m\neq\{2n-2,\,2n-1,\,2n\}}\end{aligned}$$

$$W_{nm}^{s}=\sum_{k=0}^{s-1}\sum_{p=0}^{n}\sum_{q=0}^{\varkappa}\overset{\circ}{A}_{n-p,\,m-q}A_{pq}^{s-1-k}(-\bar{R}_{12})^{k}$$

$$\varkappa=\min\{m,2p\},\quad A_{2n,\,2m}^{s}=C_{nm0}^{0s},\quad A_{2n,\,2m+1}^{s}=C_{nm0}^{1s}$$

$$A_{2n+1,\,2m}^{s}=\sum_{k=0}^{s}[(\bar{b}_{L}^{s-k}C_{nm1}^{0k}+\bar{c}^{s-k}C_{n,\,m-1,\,1}^{1k})_{m\neq 2n+1}+\bar{b}_{T}^{s-k}C_{n,\,m-1,1}^{0k}]$$

$$A_{2n+1,\,2m+1}^{s}=\sum_{k=0}^{s}[\bar{b}_{L}^{s-k}(C_{nm1}^{1k})_{m\neq 2n}+\bar{b}_{T}^{s-k}C_{n,\,m+1,1}^{1k}+\bar{c}^{s-k}C_{nm1}^{0k}]$$

$$C_{nmj}^{ks}=A_{nmj}^{ks}U_{-}(n-m-k)+B_{nmj}^{ks}U_{+}(n-m+k)$$

$$A_{nmj}^{ks}=\sum_{\alpha=0}^{m}\sum_{\beta=0}^{\alpha+k}\sum_{q=0}^{\varkappa}C_{2n+\alpha-m+j}^{m+\alpha}\,C_{2n+2\alpha-2m}^{\alpha+\beta}C_{\alpha+\beta}^{2\beta-k}\Omega_{nm\alpha\beta qk}^{1}N_{s-q,\,q},$$

$$B_{nmj}^{kw}=\sum_{\alpha=k}^{2n-m}\sum_{\beta=0}^{\alpha}\sum_{q=0}^{\eta}C_{m+\alpha+j}^{2n+m-\alpha}C_{2m+2\alpha-2n}^{2m+\alpha+\beta-2n}C_{2m+\alpha+\beta-2n}^{2\beta-k}\Omega_{nm\alpha\beta qk}^{2}N_{s-q,\,q}$$

$$\varkappa=\min\{s,2n+\alpha-m\},\quad \eta=\min\{s,m+\alpha\},\quad N_{0q}=1$$

$$N_{mq}=\frac{1}{m}\sum_{k=1}^{m}(kq-m+k)(-\bar{R}_{12})^{k}N_{m-k,\,q},\ m\geqslant 1$$

$$\Omega_{nm\alpha\beta qk}^{j}=\sum_{i=1}^{3}\sum_{q_i=q_{ij}}^{Q_{ij}}\prod_{s=1}^{3}C_{p_{sj}}^{q_j}C_{p_{4j}}^{Q_s}b_{s}^{p_{sj}-q_s}d_{1}^{p_{4j}-Q_s}\bar{b}_{s}^{q_s}\bar{d}_{2}^{Q_s},\quad Q_i=q-\sum_{j=1}^{i}q_j$$

$$q_{ij}=\max\Big\{0,Q_{i-1}-\sum_{s=i+1}^{4}p_{sj}\Big\},\quad Q_{ij}=\max\{p_{ij},Q_{i-1}\}$$

$$p_{11}=p_{12}+a_{nm},\quad p_{21}=p_{12}+k,\quad p_{31}=2\beta-k,\quad p_{41}=m-\alpha$$

$$p_{12}=\alpha-\beta,\ p_{22}=p_{21}-a_{nm},\ p_{32}=p_{31},\ p_{42}=p_{41}+a_{nm}$$

$$a_{nm}=2(n-m),\ b_1=a_L,\ b_2=a_T,\ b_3=-(b_{LT}+b_{TL})$$

$$f=f_{LT}+f_{TL},\ \bar{b}_1=\bar{e}_L,\ \bar{b}_2=\bar{e}_T,\ \bar{b}_3=-(\bar{f}_{LT}+\bar{f}_{TL})$$

$$\bar{b}_A{}^{0}=a_A,\ \bar{c}^{0}=b_3,\ \bar{b}_A{}^{m}=(-\bar{R}_{12})^{m-1}\bar{e}_A$$

$$\bar{c}_A{}^{m}=(-\bar{R}_{12})^{m-1}\bar{c}_3\ (m\geqslant 1),\ \bar{i}=\bar{i}_{LT}+\bar{i}_{TL},\ \bar{l}=\bar{l}_{LT}+\bar{l}_{TL}$$

Here $C_n{}^m$ is the binomial coefficient. Formulas (3.2)−(3.4), together with (2.10), enable us to estimate in explicit form the amplitude of any of the echo-pulses, and the times of arrival at the observer, which can be found from the equation $\tau=L_{nms}{}^{+}=\text{const}$ (3.4).

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DIFFRACTION OF A PLANE WAVE BY AN INFINITE ELASTIC PLATE STIFFENED BY A DOUBLY PERIODIC SET OF RIGID RIBS*

B. P. BELINSKII

The diffraction of a plane wave by an infinite elastic plate stiffened by a doubly periodic set of rigid ribs of moderate wave dimensions is studied. The problem is reduced to an infinite quasiregular system of linear algebraic equations, and their solution describes the amplitudes of the waves propagating from the plate into the fluid.

The effect of a periodic set of parallel ribs, which stiffen an elastic plate, on its acoustic properties, has been studied in reasonable detail. An exact solution of the problem of the diffraction of a plane wave by such a plate is given in /1/ where the frequency relationships of the reflection and transmission coefficients of a plane wave were also studied and simple approximate formulas were obtained for the limiting cases.

1. We will investigate the diffraction of a plane pressure wave

$$p_0 = \exp(ik((x\cos\varphi_0 + y\sin\varphi_0)\sin\theta_0 - z\cos\theta_0))$$

incident on an infinite plate $\{-\infty < x, y < \infty, z = 0\}$ stiffened by a doubly periodic set of rigid ribs $\{-\infty < x < \infty, y = mb; -\infty < y < \infty, x = na; -\infty < n, m < \infty\}$. The pressure $p(x, y, z)$ in the medium satisfies the Helmholtz equation with the boundary condition at the plate given by

$$D(\Delta_0^2 - k_0^4)\xi(x, y) + [p](z = 0) = 0 \tag{1.1}$$

$$(x \neq na,\ y \neq mb)$$

$$k_0 = (\rho^\circ\omega^2 H^\circ/D)^{1/4}$$

Here D is the cylindrical rigidity of the plate, $\xi(x, y)$ is its displacement, connected with the pressure by the adhesion condition $\xi(x, y) = p_z(x, y, 0)/(\rho_0\omega^2)$, ρ_0 is the density of the liquid, Δ_0 is the two-dimensional Laplace operator, k_0 is the wave number of the flexural waves in the plate, ρ° is the plate density and H° is its thickness. The symbol $[\varphi](z = 0)$ denotes the jump in the value of the function φ at $z = 0$. The harmonic dependence of the processes on time $\exp(-i\omega t)$ is omitted.

We will first assume that fluid is present on one side of the plate only $(z > 0)$. The case of two-sided contact can be studied in exactly the same manner. We shall therefore only refer to it at the stage of numerical analysis and interpretation of the results. The boundary contact conditions (BCC) appear when the bending and torsional oscillations of the ribs and their rigid coupling to the plate carrying them are taken into account /2/

$$-D[\xi_{xxx} + (2 - \sigma)\xi_{xyy}](x = na) = -i\omega Z_{11}\xi \tag{1.2}$$

$$D[\xi_{xx} + \sigma\xi_{yy}](x = na) = -i\omega Z_{12}\xi_x$$

$$(x = na,\ y \neq mb)$$

$$-D[\xi_{yyy} + (2 - \sigma)\xi_{yxx}](y = mb) = -i\omega Z_{21}\xi$$

$$D[\xi_{yy} + \sigma\xi_{xx}](y = mb) = -i\omega Z_{22}\xi_y \quad (y = mb,\ x \neq na)$$

Here σ is Poisson's ratio of the plate, and the operators Z_{p1}, Z_{p2} $(p = 1, 2)$ are the force and momentum impedances of the ribs respectively.

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